Exam Seat No:________________ **Enrollment No:____________________**

C.U.SHAH UNIVERSITY

WADHWAN CITY

University (Winter) Examination -2013

Course Name :M.Sc(Mathematic) Sem-I **Subject Name: -Linear Algebra**

Duration :- 3:00 Hours **Date : 16/12/2013**

Instructions:-

(1) Attempt all Questions of both sections in same answer book / Supplementary.
(2) Use of Programmable calculator & any other electronic instrument is prohibited.
(3) Instructions written on main answer Book are strictly to be obeyed.
(4) Draw neat diagrams & figures (If necessary) at right places.
(5) Assume suitable & Perfect data if needed.

SECTION-I

Q-1 a) Define: Vector Space. (02)

b) Is $\{x, \cos x\}$ linearly independent ? (01)

c) Prove that $\dim R^n$ over R is n. (01)

d) Let V be a vector space over F and S be non-empty subset of V then prove that $L(S)$ is subspace of V. (02)

e) Define $T: R^2 \to R^3$, $T(x,\ y) = (x,\ x+3y,\ 2y)$. Is T linear ? (01)

Q-2 a) Show that R^+ is a vector space with the operations defined as $x + y = xy$, and $kx = x^k$. (05)

b) Let V be a finite dimensional vector space over F and $v_1, v_2, \dots, v_k$ be linearly independent vectors in V. Show that there are vectors $v_{k+1}, v_{k+2}, \dots, v_n$ in V such that $v_1, v_2, \dots, v_k, v_{k+1}, v_{k+2}, \dots, v_n$ is a basis of V. (05)

c) Determine whether $\{1 + x,\ 1 - x,\ x^2\}$ is linearly independent. (04)

OR

Q-2 a) Determine which of the following are subspaces of M_{22}. (05)

(i) all 2×2 matrices with integer entries.

(ii) all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a + b + c + d = 0$.

b) If A is an algebra, with unit element, over F, then prove that A is isomorphic to a subalgebra of $A(V)$ for some vector space V over F. (05)

c) Find the coordinate vector of $v = (5, -12,\ 3)$ relative to the basis $S = \{v_1, v_2, v_3\}$ where $v_1 = (1,\ 2,\ 3)$, $v_2 = (-4,\ 5,\ 6)$, $v_3 = (7, -8,\ 9)$. (04)

Q-3 a) Let V be a finite dimensional vector space over F and W be a subspace of V, then prove that W is a finite dimensional and $\dim W \le \dim V$. Also (05)

$$\dim {}^{V}\!/_{W} = dim\, V - dimW.$$

b) Let V and W be vector spaces over F and $\emptyset : V \to W$ be a homomorphism, then prove that ${}^{V}\!/_{\ker \emptyset}$ is isomorphic to W. (05)

c) If $v_1, v_2, \dots, v_n$ are in a vector space V then prove that either they are linearly independent or some v_k is a linear combination of the preceding ones, $v_1, v_2, \dots, v_{k-1}$. (04)

OR

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Q-3 a) Let V and W be finite dimensional vector space over F, then prove that the set Hom $(V, W) = \{T: V \to W; T \text{ is homomorphism}\}$ is also finite dimensional vector space and $\dim Hom\ (V,\ W) = \dim V \cdot \dim W$. (05)

b) Let V be a finite dimensional vector space over F and W be a subspace of V, then prove that $\widehat{W}$is isomorphic to $\hat{V}\ /\ W^o$. (05)

c) If $v_1, v_2, \ldots, v_n$ is a basis of V over F and if $w_1, w_2, \ldots, w_m$ in V are linearly independent over F, then prove that $m \le n$. (04)

SECTION-II

Q-4 a) For any $n \ge 1$, the determinant of the identity matrix I_n is 1. (02)

b) Let $A, B \in M_n(F)$. Show that $tr(A + B) = tr(A) + tr(B)$. (02)

c) Define: Nilpotent. (01)

d) Define determinant of any $n \times n$ matrix. (01)

e) Let $A \in M_n(F)$ be nilpotent. What is the $tr(A)$? (01)

Q-5 a) Let V be a finite dimensional vector space over F and $T \in A(V)$ be such that all characteristic roots are in F, then prove that there exists a basis of V in which the matrix is triangular. (05)

b) Let V be a finite dimensional vector space over, $T \in A(V)$ and W be a subspace on V invariant under T. Define the linear transformation $\bar{T}$of T on $\bar{V} = {}^{V}\!/_{W}$. Suppose $p(x)$ and $p_1(x)$are minimal polynomial for $T$and $\bar{T}$ respectively, then prove that $p_1(x) \big/ p(x)$. (05)

c) Let V be a finite dimensional vector space over. If F be a field of characteristic 0 and $T \in A(V)$ is such that $tr\ T^i = 0\ \ for\ i = 1,2, \ldots$, then prove that T is nilpotent (04)

OR

Q-5 a) Let V be an n-dimensional vector space over F and $T \in A(V)$ be such that all characteristic roots of T are in F. Show that there is a polynomial $p(x) \in F[x]$ of degree n such that $p(T) = 0$. (05)

b) If $\lambda \in F$ is a characteristic root of $T \in A(V)$, then prove that for any polynomial $q(x) \in F[x]$, $q(\lambda)$ is a characteristic root of $q(T)$. (05)

c) Let F be a field of characteristic 0 and V be a finite dimensional vector space over F. Let $S, T \in A(V)$ be such that $(ST - TS)$is commutes with S, then show that $(ST - TS)$ is nilpotent. (04)

Q-6 a) Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent. Show that the invariants of T are unique. (05)

b) If V is finite dimensional over F and if $T \in A(V)$ is singular, then prove that there exist an $S \ne 0$ in $A(V)$ such that $ST = TS = 0$. (05)

c) Let$A, B \in M_n(F)$. Show that $\det(A) = \det(A)\det(B)$. (04)

OR

Q-6 a) Suppose that $V = V_1 \oplus V_2$, where V_1 and V_2 are subspaces of V invariant under T. Let T_1 and T_2 be the linear transformations induced by T on V_1 and V_2, respectively. If the minimal polynomial for T_1 over F is $p_1(x)$ while that of T_2 is $p_2(x)$, then prove that the minimal polynomial for Tover F is the least common multiple of $p_1(x)$ and $p_2(x)$. (05)

b) Let $T \in A_F(V)$ have all its distinct characteristic roots, $\lambda_1, \lambda_2, \dots, \lambda_k$ in F. Then prove that a basis for V can be found in which the matrix T is of the form (05)

$$\begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}.$$

c) A is invertible if and only if $\det A \neq 0$. (04)

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